International Journal of Theoretical Physics, Vol. 35, No. 10, 1996

Wave Functions in Geometric Quantization

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Received August 25, 1995

A geometrical way is described to associate quantum states in the sense of geometric quantization to wave functions in the quantum mechanical sense for each relativistic elementary particle. Explicit computations are made in a number of cases: Klein–Gordon and Dirac equations, neutrino and antineutrino Weyl equations, and very general cases of massive and massless particles of arbitrary spin. In this later case one is led in a canonical way to Penrose wave equations.

1. INTRODUCTION

The wave equations of relativistic elementary particles, Klein–Gordon, Dirac, Maxwell, etc., were originally each derived independently. A unification resulted through the discovery of the relation of these equations with the representations of the Poincaré group (inhomogeneous Lorentz group). The classification of the representations of the Poincaré group made by Wigner (1939) with important contributions of Majorana (1932), Dirac (1936), and Proca (1936) led to the group-theoretic study of wave equations by Bargmann and Wigner (1948).

In Kirillov–Kostant–Souriau theory (geometric quantization) the description of quantum system is given in terms of elements of the dual of the Lie algebra of the Lie group under consideration. Of course this way of seeing quantum mechanics is not completely independent of the preceding one, since it has its origin in a method of obtaining representations (Kirillov, 1962; Auslander and Kostant, 1971). But the correspondence of quantum states in the sense of geometric quantization to wave functions in the quantum mechanical sense is not clear in all cases. Souriau (1970) gives a very general way to make the passage, but it does not work in all cases. For example, in

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0020-7748/96/1000-2139$09.50/0

the relativistic case, not every quantum state for massive particles with 1/2 spin corresponds to a solution of the Dirac equation.

In this paper we give a geometrical construction that establishes a one-to-one correspondence from quantum states in the sense of geometric quantization to wave functions in the quantum mechanical sense that is valid for all kinds of relativistic elementary particles. This solves in particular the preceding problem. The idea is as follows.

In geometric quantization (GQ) one begins with a regular contact manifold or its associated Hermitian line bundle. Quantum states (in the sense of geometric quantification) can be considered as being the collection of those sections of the Hermitian line bundle which satisfy "Planck's condition" (Souriau, 1970). In this paper we see that these sections are in a one-to-one correspondence with the (unrestricted) sections of another Hermitian line bundle. Thus, this fiber bundle is a good setting to describe the quantum processes under consideration. The main idea for passing from this description to the usual one in terms of wave functions can be intuitively explained as follows. Quantum states in GQ attribute an "amplitude of probability" to each movement of the particle. To obtain the corresponding wave function one must proceed as follows: for each event, the corresponding amplitude of probability is obtained by taking all movements passing through the given event and then "adding up" (in a suitable sense) the corresponding amplitudes of probability. Of course, the concept of "movement passing through an event" is only obvious in the case of the ordinary massive spinless particle, and is defined in Section 3.

In the case of massive particles one obtains solutions of Klein–Gordon and Dirac equations, and also a description of the wave functions for massive particles of higher spin. In the case of massless particles of spin 1/2 one obtains solutions of the Weyl equations and for general spin this method leads in a natural way to the description of massless particles by means of solutions of the Penrose wave equations (Penrose, 1975).

2. UNIVERSAL COVERING GROUP OF POINCARÉ GROUP

It is well known that the universal covering group of the Poincaré group is a semidirect product of **SL(2, C)** by a four-dimensional real vector space. In this section, I recall some general facts about this group.

Let (x^1, x^2, x^3, x^4) be the canonical coordinates in $\mathbf{R}^4$, $\mathbf{I}$ the 2×2 unit matrix, and $\sigma_1, \sigma_2, \sigma_3$ the Pauli matrices, i.e.,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

respectively. A generic point of $\mathbf{R}^4$ will be denoted by $x = (x^1, x^2, x^3, x^4)$.

We define an isomorphism h from $\mathbf{R}^4$ onto the real vector space $\mathbf{H}(2)$ of the Hermitian 2×2 matrices by means of

$$h(x) = x^4 I + \sum_{i=1}^{3} x^i \sigma_i = \begin{pmatrix} x^4 + x^3, & x^1 - ix^2 \\ x^1 + ix^2, & x^4 - x^3 \end{pmatrix}$$

We have Det $h(x) = \langle x, d \rangle$, where $\langle , \rangle$ is Minkowski pseudoscalar product

$$\langle x, y \rangle = x^4 y^4 - \sum_{i=1}^{3} x^i y^i$$

We define an action on the left of the Lie group $\mathbf{SL(2, C)}$ on the Abelian Lie group $\mathbf{H(2)}$ by means of

$$A * H = AHA^*$$

for all $A \in \mathbf{SL(2, C)}$, $H \in \mathbf{H(2)}$, where A^* is the transpose of the complex conjugate of A. To this action by automorphisms of $\mathbf{H(2)}$ there corresponds a semidirect product, $\mathbf{SL(2, C)} \oplus \mathbf{H(2)}$, whose group law is given by

$$(A, H) * (B, K) = (AB, AKA^* + H)$$

The identity element is $(I, 0)$ and $(A, H)^{-1} = (A^{-1}, -A^{-1}HA^{*-1})$.

This semidirect product acts on the left on $\mathbf{R}^4$ by means of

$$(A, H) * x = h^{-1}(Ah(x)A^* + H)$$

The Poincaré group $\mathcal{P}$ is identified with the closed subgroup of $GL(5; \mathbf{R})$ composed of the matrices

$$\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}$$

where $C \in \mathbf{R}^4$ and $L \in \mathbb{O}(3, 1)$ [such a matrix is denoted in the following simply by (L, C)].

For all $(A, H) \in \mathbf{SL} \oplus \mathbf{H(2)}$ [where $\mathbf{SL}$ stands for $\mathbf{SL(2; C)}$], there exists a unique $(L, C) \in \mathcal{P}$ such that $(A, H) * x = Lx + C$ for all $x \in \mathbf{R}^4$.

The map ρ from $\mathbf{SL} \oplus \mathbf{H(2)}$ into $\mathcal{P}$ defined by sending such an (A, H) to the corresponding (L, C) is a homomorphism of Lie groups whose kernel consists of $(I, 0)$ and $(-I, 0)$. Since both Lie groups have the same dimension, ρ is a covering map of the identity component in $\mathcal{P}$, $\mathcal{P}_+^\uparrow$. Since $\mathbf{SL}$ and $\mathbf{H(2)}$ are connected and simply connected, it follows that $\mathbf{SL} \oplus \mathbf{H(2)}$ is the universal covering group of $\mathcal{P}_+^\uparrow$.

The standard method to handle semidirect products enable us to identify the Lie algebra of $\mathbf{SL} \oplus \mathbf{H(2)}$ with $\mathbf{sl} \times \mathbf{H(2)}$, the Lie bracket being

$$[(a, k), (a', k')] = ([a, a'], ak' + k'a^* - (a'k + ka'^*))$$

In this paper, we use the basis of $\mathbf{sl} \times \mathbf{H(2)}$ composed of the following elements:

$$P^k = (0, -\sigma_k), \qquad k = 1, 2, 3$$
$$P^4 = (0, \sigma_4) = (0, I)$$
$$l^k = \left(i\frac{\sigma_k}{2}, 0\right)$$
$$g^k = \left(\frac{\sigma_k}{2}, 0\right)$$

The reason for this notation will become clearer in the next section.

We denote by ϵ the matrix $i\sigma_2$,

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Notice that ${}^t\!A\epsilon A = (\mathrm{Det}\, A)\epsilon$, so that

$$\epsilon\, {}^t\!A \epsilon = -A^{-1} \tag{1}$$

if $A \in \mathbf{SL}$. Also we have

$$\tfrac{1}{2}\,\mathrm{Tr}(h(x)\epsilon\overline{h(y)}\epsilon) = -\langle x, y\rangle$$

for all $x, y \in \mathbf{R}^4$, where the bar means complex conjugation.

We define a nondegenerate scalar product in $\mathbf{sl} \times \mathbf{H(2)}$ by means of

$$\langle (a, k), (b, l)\rangle = 2\,\mathrm{Re}\,\mathrm{Tr}(\tfrac{1}{4}k\epsilon\bar{l}\epsilon - ab)$$
$$= \tfrac{1}{2}\,\mathrm{Tr}(k\epsilon\bar{l}\epsilon) - 2\,\mathrm{Re}\,\mathrm{Tr}\, ab$$

This scalar product defines in the standard way an isomorphism from the Lie algebra of $\mathbf{SL} \oplus \mathbf{H(2)}$ onto its dual. The image of $(a, k) \in \mathbf{sl} \times \mathbf{H(2)}$ will be denoted by $\{a, k\}$.

With this notation, we obtain by a more or less straightforward computation the following formula for the coadjoint representation (i.e., the contragredient of the adjoint representation):

$$\mathrm{Ad}^*_{(A,H)}\{a, k\} = \{AaA^{-1} + \tfrac{1}{4}(AkA^*\epsilon\overline{H}\epsilon - H\epsilon\overline{AkA^*}\epsilon), AkA^*\} \tag{2}$$

The elements of the Lie algebra define functions on its dual in the well-known way. This is in particular the case of P^j, l^i, g^i and we have

$$h(P(\{a, k\})) = -k$$
$$h(\vec{l}(\{a, k\}), 0) = i(a^* - a)$$
$$h(\vec{g}(\{a, k\}), 0) = -(a + a^*)$$

where $P = (P^1, P^2, P^3, P^4)$, $\vec{l} = (l^1, l^2, l^3)$, and $\vec{g} = (g^1, g^2, g^3)$.

To end this section, we define two other functions on the dual of the Lie algebra of **SL** ⊕ **H(2)** whose physical meaning will become clearer in the next section. The main interest of these functions, at least from the group-theoretic point of view, is that they remain constant along coadjoint orbits.

One of these is $|P|$, defined by $|P|(\{a, k\}) = \mathrm{Det}(k)$. The other is defined in terms of $W(\{a, k\}) = i(ak - ka^*)$. One can prove that $W(\mathrm{Ad}^*_{(A,H)}\{a, k\}) = AW(\{a, k\})A^*$, so that the function $|W|(\{a, k\}) = \mathrm{Det}(W(\{a, k\}))$ is constant along each coadjoint orbit.

3. CLASSICAL STATE SPACE

In this section we recall some known results and we establish some of the definitions and interpretations that are necessary for the purposes of this paper. Most of the known results can be found in Souriau (1970) in relation to the Poincaré group. We use, instead of this group, the universal covering group of its components of the identity $G =$ **SL(2, C)** ⊕ **H(2)**. The results remain obviously valid.

Let us consider a classical (without spin) relativistic free particle with rest mass $m \neq 0$. A classical state of this particle is given in each given inertial frame by the coordinates of an event and a value of momentum-energy. The set composed by these classical states (state space or evolution space) can be identified with a codimension-1 submanifold of TM (or T^*M), where M is Minkowski space-time. G acts transitively on the left on state space.

The momentum map sends the state space onto a coadjoint orbit of G in a G-equivariant way. Each classical movement of this particle is composed by a set of classical states and the momentum map sends all of these states to the same point of the orbit in such a way that no other state is mapped to this point. In this way, the momentum map establishes a one-to-one map from movements of the particle and a coadjoint orbit. Thus the coadjoint orbit is identified with "movement space." The momentum map, accompanied by the projection on Minkowski space-time, gives, for each inertial observer, a G-equivariant imbedding of the state space into $H(2) \times \underline{G}^*$. This enables us to identify the state space with an orbit of G in $H(2) \times \underline{G}^*$. When this identification is done the momentum map becomes the canonical map of an orbit in $H(2) \times \underline{G}^*$ onto the corresponding orbit in $\underline{G}^*$.

The elements P^i, l^k, g^k (see Section 2) of $\underline{G}$ give rise, via the momentum map, to dynamical variables on state space, whose expression in each inertial frame is the classical one for momentum, energy, and relativistic angular momentum ($c = 1$).

In the general case of particles with spin or massless particles, the concept of state space is much less clear *a priori*. Souriau defined its "evolution spaces" as candidates. All of them are easily seen to be diffeomorphic to orbits of G in $H(2) \times \underline{G}^*$. If one takes into account the possibility of changes of inertial frames, one is led *a priori* to consider as possible *classical state spaces* all the orbits of G in $H(2) \times \underline{G}^*$. Thus, each classical state space projects onto a coadjoint orbit that is interpreted as the corresponding *movement space*. A key fact of quantum theory is that we must consider as possible movement spaces only the coadjoint orbits that are quantizable in the sense of the next section.

With this definition, the classical state space of a particle whose movement space is a given quantizable coadjoint orbit is to some extent undefined: there are infinitely many candidates. We shall see in the next section that the requirement that quantum mechanics be independent of the choice of classical state space leads to one of the essential axioms of geometric quantization: Planck's condition.

When a concrete classical state space has been fixed, one can ask questions such as: Which is the family of events which represents a given movement in space-time? Which are the movements whose representation in space-time contains a given event? In order to explain this in more detail, let us consider an elementary particle whose movement space is the coadjoint orbit of $\alpha \in \underline{G}^*$ and let us choose its classical state space to be the orbit by G of (K, α) in $H(2) \times \underline{G}^*$, where $K \in H(2)$. The movement $\mathrm{Ad}^*_{(A,H)}\alpha$ is represented in space-time by the events $L \in H(2)$ such that $(L, \mathrm{Ad}^*_{(A,H)}\alpha)$ is in state space, i.e., the orbit in space-time of K by $(A, H) \cdot G_\alpha$, where G_α is the isotropy subgroup of α. The movements passing through an event H are the β such that (H, β) is in state space. These β can also be easily characterized group theoretically. More specific computations are made in Section 5.

The functions defined on each classical state space by P^i, l^k, g^k are interpreted as giving momentum, energy, and angular momentum and are denoted by the same letter. With this interpretation, the mass square coincides with the function $|P|$ defined in the preceding section.

The Pauli–Lubanski four-vector corresponds to $(P^4\vec{l} + \vec{p} \times \vec{g}, \langle \vec{p}, \vec{l} \rangle)$, where $\vec{p} = (P^1, P^2, P^3)$. A more or less straightforward computation proves that the Hermitian matrix corresponding to this four-vector coincides with the W defined in Section 2.

4. QUANTUM STATES

In this section, we use a definition of quantum states of elementary particles that is equivalent to that of Souriau (1970, 1988), with a slightly different notation. We also prove some results that enable us to identify quantum states with unrestricted sections of a Hermitian line bundle instead of sections of another Hermitian line bundle subjected to the "Planck condition" of Souriau.

In order to carry out the geometric quantization of a symplectic manifold, one first needs a Boothby–Wang fibration on the symplectic manifold with circles as fibers, i.e., a principal circle bundle with connection whose curvature form (projected on the base space) is the symplectic form.

Here we shall consider only the case where the symplectic manifold is the movement space of a relativistic elementary particle, i.e., a coadjoint orbit of the universal covering group of the Poincaré group, and the contact manifold is a homogeneous contact, for an action of the same group, such that the bundle projection becomes equivariant.

In order to explain in more detail the geometric constructions we need, let us recall some results concerning the homogeneous contact manifolds under consideration. These results and a study of more general situations can be found in Souriau (1988) and Díaz Miranda (1982a,b, n.d.). Some of these generalizations also have interest from the point of view of the present paper. In fact, one can consider each covering of a coadjoint orbit as a candidate for movement space, and the geometric construction that follows would remain valid. But for the purposes of this paper, it is enough to consider the coadjoint orbits themselves. In all that concerns fiber bundles, we use the notation of Kobayashi and Nomizu (1963).

Let G be a Lie group. A fibration as desired on the coadjoint orbit of $\alpha \in \underline{G}^*$ exists if and only if there exists a surjective homomorphism C_α from the isotropy subgroup at α, G_α, onto the unit circle $\mathbf{S}^1$ whose differential is α. Then α and its coadjoint orbit are said to be *quantizable*. Here, we can consider the differential of a homomorphism onto $\mathbf{S}^1$ as an ordinary 1-form by identifying the Lie algebra of $\mathbf{S}^1$ with $\mathbf{R}$. This identification is defined by the condition that the exponential map becomes $\mathrm{Exp}(a) = e^{2\pi i a}$ for all $a \in \mathbf{R}$. The α and its coadjoint orbit are said to be $\mathbf{R}$-*quantizable* if there exists a surjective homomorphism from G_α onto $\mathbf{R}$ whose differential is α. The Lie algebra of $\mathbf{R}$ is identified with $\mathbf{R}$ in such a way that the exponential map becomes the identity. Of course, if α is $\mathbf{R}$-*quantizable* it is *quantizable*. Díaz-Miranda (n.d.) uses a slightly more general concept of quantizability, but it is unnecessary for the purposes of the present paper.

In what follows we assume that α is quantizable and C_α is a homomorphism from G_α onto the unit circle whose differential is α. We identify the coadjoint orbit with G/G_α in the canonical way.

We define an action of $\mathbf{S}^1$ on $G/\mathrm{Ker}C_\alpha$ by means of

$$(g\ \mathrm{Ker}C_\alpha) * s = gh\ \mathrm{Ker}C_\alpha \tag{3}$$

where h is any element of G_α such that $C_\alpha(h) = s$. Actually $(G/\mathrm{Ker}C_\alpha)$ $(G/C_\alpha, \mathbf{S}^1)$ is a principal fiber bundle, the bundle action being the preceding one and the bundle projection being the canonical map from $G/\mathrm{Ker}C_\alpha$ onto G/G_α.

The differential 1-form α projects to an invariant contact form ω on $G/\mathrm{Ker}C_\alpha$.

Let $Z(\omega)$ be the vector field defined by $i_{Z(\omega)}\omega = 1$, $i_{Z(\omega)}\, d\omega = 0$. All the integral curves of $Z(\omega)$ have the same period. If we denote by $T(\omega)$ the period of any integral curve of $Z(\omega)$, then $\omega/T(\omega)$ is a connection form. Since the structural group is Abelian, the curvature form is $d\omega/T(\omega)$. There exists a unique 2-form on G/G_α whose pullback under the bundle map is the curvature form. This form is symplectic and its cohomology class is integral. It will also be called a curvature form. Its reciprocal image under the canonical map of G onto G/G_α is $d\alpha/T(\omega)$. These symplectic manifolds and their covering spaces are Hamiltonian spaces of the group G (Kostant, 1970).

The horizontal lift of curves can be described as follows. Given a curve γ in G/G_α, the horizontal lift of γ to $g\ \mathrm{Ker}C_\alpha$ is

$$\tilde{\gamma}(t) = (\bar{\gamma}(t)\ \mathrm{Ker}C_\alpha) * \exp\left(-2\pi i \int_{\bar{\gamma}\,|[0,t]} \alpha\right) \tag{4}$$

where $\bar{\gamma}$ is any lifting of γ to G such that $\bar{\gamma}(0) = g$, and the vertical bar means restriction.

Associated to this principal fiber bundle and the canonical action of $\mathbf{S}^1$ on $\mathbf{C}$, one can consider the one-dimensional vector bundle whose total space is $(G/\mathrm{Ker}C_\alpha) \times_{\mathbf{S}^1} \mathbf{C}$. This bundle is a complex line bundle; the addition in each fiber is given by

$$[g\ \mathrm{Ker}C_\alpha, z] + [g'\ \mathrm{Ker}C_\alpha, z'] = [g\ \mathrm{Ker}C_\alpha, z + C_\alpha(g^{-1}g')z']$$

and the multiplication by complex numbers is given by $a \cdot [g\ \mathrm{Ker}C_\alpha, z] = [g\ \mathrm{Ker}C_\alpha, az]$.

This vector bundle becomes Hermitian when one defines in each fiber the Hermitian product

$$\langle [g\ \mathrm{Ker}C_\alpha, z], [g'\ \mathrm{Ker}C_\alpha, z'] \rangle = \bar{z}C_\alpha(g^{-1}g')z'$$

It is well known that the sections of the Hermitian line bundle are in one-to-one correspondence with the functions on $G/\mathrm{Ker}C_\alpha$, f, such that $f((g\ \mathrm{Ker}C_\alpha) * s) = s^{-1}f(g\ \mathrm{Ker}C_\alpha)$. These functions will be called *pseudotensorial functions*. This correspondence is as follows. If f is a pseudotensorial function, the corresponding section sends $m \in G/G_\alpha$ to $[r, f(r)]$, where r is arbitrary in the fiber on m. If σ is a section of the Hermitian line bundle, the corresponding pseudotensorial function f is defined by $\sigma(\pi(r)) = [r, f(r)]$ for all $r \in G/\mathrm{Ker}C_\alpha$, where π is the bundle projection.

The sections of the Hermitian line bundle are called *prequantum states*. Sometimes we use the same denomination for the corresponding pseudotensorial functions.

Let us return to the case where $G = \mathbf{SL(2, C)} \oplus \mathbf{H(2)}$. As we saw in Section 3, there are many candidates for state space for the particle whose movement space is the orbit of α. In fact, for each $K \in H(2)$, the orbit of $(K, \alpha) \in H(2) \times \underline{G}^*$ is one of them.

If state space is the orbit of (K, α) and $H \in H(2)$, the movement containing the event H are the $\mathrm{Ad}^*_{(A,L)}\alpha$ such that $(A, L) * K = H$, i.e., $A \in SL(2, C)$ and $L = H - AKA^*$. This set depends on the choice of K.

We call quantum states those prequantum states that are independent of the preceding choice in the following sense.

Since $\mathrm{Ad}^*_{(A,H-AKA^*)}\alpha = \mathrm{Ad}^*_{(I,-AKA^*)}\mathrm{Ad}^*_{(A,H)}\alpha$, we say that a prequantum state, considered as a section of the Hermitian line bundle, is independent of the choice of K if this value on the right-hand side of the preceding equation is independent, up to parallel transport, of the actual value of K, i.e., if $\phi(\mathrm{Ad}^*_{(I,L)}\gamma) = \tau(\phi(\gamma))$ for all γ in the orbit and L in $H(2)$, where τ is parallel transport along any curve joining γ with $\mathrm{Ad}^*_{(I,L)}\gamma$ in the orbit of γ by the subgroup $\{I\} \times H(2)$. An equivalent statement of this condition is that the corresponding pseudotensorial function is constant along the horizontal lift of such a curve; thus we have the following.

Definition 4.1. A *quantum state* is a prequantum state whose corresponding pseudotensorial function is constant along the horizontal lift of any curve whose image is in an orbit of the subgroup $\{I\} \times H(2)$.

Lemma 4.1. There exists a unique action of $\{I\} \times H(2)$ on $G/\mathrm{Ker}C_\alpha$ whose orbits are horizontal and such that π becomes equivariant. This action is given by

$$(I, K) * ((A, H)\ \mathrm{Ker}C_\alpha) = ((A, H + K)\ \mathrm{Ker}C_\alpha) * \exp[-i\pi\ \mathrm{Tr}(AkA^*\epsilon\overline{K}\epsilon)] \tag{5}$$

for all $K \in H(2)$, $(A, H) \in G$, where $*$ on the left-hand side stands for the new action and on the right-hand side corresponds to the bundle action. k is given by $\alpha = \{a, k\}$.

Proof. Let us denote $\{I\} \times H(2)$ simply by H. If $X \in \underline{H}$, we denote by X_b the corresponding infinitesimal generator of the action on G/G_α and by X^*_b its horizontal lift to $G/\mathrm{Ker}C_\alpha$. Since the integral curves of X^*_b are horizontal lifts of integral curves of X_b, we see that X^*_b is a complete vector field.

We shall prove that $[X^*_b, Y^*_b] = 0$ for all $X, Y \in \underline{H}$.

First notice that $[X^*_b, Y^*_b]$ is horizontal. In fact, since X^*_b and Y^*_b are horizontal, we have $\omega([X^*_b, Y^*_b]) = -d\omega(X^*_b, Y^*_b)$. Since $d\omega$ projects to the symplectic form Ω of G/G_α given by the projection under the canonical map of the 2-form of G, $d\alpha$, we have

$$\begin{aligned}&(d\omega(X_b^*, Y_b^*))(g\ \mathrm{Ker}C_\alpha)\\ &= (\Omega(X_b, Y_b))(g\ G_\alpha)\\ &= (d\alpha)(\mathrm{Ad}_{g^{-1}}X, \mathrm{Ad}_{g^{-1}}Y) = -\alpha([\mathrm{Ad}_{g^{-1}}X, \mathrm{Ad}_{g^{-1}}Y])\end{aligned}$$

which vanishes since H is Abelian.

Thus, $[X_b^*, Y_b^*]$ coincides with its horizontal part, but the horizontal part of $[X_b^*, Y_b^*]$ is the horizontal lift of $[X_b, Y_b]$ and H is Abelian. It follows that $[X_b^*, Y_b^*]$ is zero.

As a consequence, the set composed of the X_b^*, $X \in \underline{H}$, is an Abelian Lie algebra of complete vector fields on $G/\mathrm{Ker}C_\alpha$. Since H is simply connected, there exists a unique action of H on $G/\mathrm{Ker}C_\alpha$ whose infinitesimal generators are the X_b^*. The bundle map is equivariant for this action and the canonical one is the base space as a consequence of the fact that the infinitesimal generators X_b^* projects onto the corresponding ones X_b, and H is connected.

Unicity follows from the fact that the infinitesimal generators of such an action must be the X_b^*.

Now, let us prove equation (5). Let $K \in H(2)$, $(A, H) \in G$. We consider the curve in G given by $\overline{\gamma}(t) = (I, tK)(A, H)$, the curve in G/G_α given by $\gamma(t) = \overline{\gamma}(t)G_\alpha$, and the curve in $G/\mathrm{Ker}C_\alpha$ given by $\rho(t) = (I, tK) * ((A, H)\ \mathrm{Ker}C_\alpha)$, $t \in [0, 1]$, where $*$ stands for the action whose existence we have just proved. $\rho(t)$ is contained in the orbit of $(A, H)\ \mathrm{Ker}C_\alpha$ by $I \times H(2)$, so that it is horizontal, and projects onto $(I, tK)\cdot((A, H)G_\alpha) = \gamma(t)$. Thus, $\rho(t)$ is the horizontal lift $\tilde{\gamma}(t)$ of $\gamma(t)$ to $(A, H)\ \mathrm{Ker}C_\alpha$, so that it is given by equation (4). Since the left-hand member of (5) coincides with $\rho(1)$, we only need to prove that

$$\int_{\overline{\gamma}} \alpha = (1/2)\ \mathrm{Tr}(AkA^*\epsilon\overline{K}\epsilon)$$

A direct computation proves that

$$\int_{\overline{\gamma}} \alpha = (1/2)\ \mathrm{Tr}(k\epsilon\overline{A^{-1}KA^{*-1}}\epsilon)$$

and thus the result is a consequence of equation (1). ∎

This action will be called the *horizontal action*.

Corollary 4.1. The quantum states are the prequantum states that correspond to pseudotensorial functions left invariant by the horizontal action.

The condition of being invariant by the horizontal action is equivalent to the Planck condition of Souriau.

We shall see later that the canonical action of G on $G/\mathrm{Ker}C_\alpha$ maps horizontal orbits to horizontal orbits, thus giving a transitive action on the set consisting of these submanifolds. In order to describe the isotropy subgroup at the orbit of $\mathrm{Ker}C_\alpha$, we define two homomorphisms as follows.

Let π_1 and π_2 be the canonical projections of G on $SL(2, \mathbf{C})$ and $H(2)$, respectively. We denote $\pi_1(G_\alpha)$ by $(G_\alpha)_{SL}$ and $\pi_2(G_\alpha)$ by $(G_\alpha)_H$.

Lemma 4.2. The map $(C_\alpha)_{SL}$: $(G_\alpha)_{SL} \mapsto S^1$ defined by

$$(C_\alpha)_{SL}(g) = C_\alpha(g, h)e^{-i\pi\mathrm{Tr}(k\epsilon\bar{h}\epsilon)} \qquad \text{for all} \quad (g, h) \in G_\alpha$$

is well defined and a homomorphism.

Proof. Let $h, h' \in H(2)$ be such that $(a, h), (a, h') \in G_\alpha$. Then $(I, h - h') \in G_\alpha$, since it coincides with $(a, h)(a, h')^{-1}$.

But we have $(I, h) \in G_\alpha$ if and only if $k\epsilon\bar{h}\epsilon = h\epsilon\bar{k}\epsilon$. Thus, if $(I, h) \in G_\alpha$, the same holds for (I, th) for all $t \in \mathbf{R}$. As a consequence, (I, h) is in the connected component of the identity G_α^0 of G_α. Hence $(I, h - h') \in G_\alpha^0$.

Since the differential of C_α is α, we have

$$C_\alpha(I, t(h - h')) = e^{2\pi it\alpha((0,h-h'))}$$

Thus

$$e^{\pi i\mathrm{Tr}(k\epsilon\bar{h}\epsilon)}e^{-\pi i\mathrm{Tr}(k\epsilon\bar{h}'\epsilon)} = C_\alpha(a, h)\cdot(C_\alpha(a, h'))^{-1}$$

which proves that $(C_\alpha)_{SL}$ is well defined. It can be verified directly that it is a homomorphism. ∎

We also define $\tilde{C}_\alpha$: $(G_\alpha)_{SL} \oplus H(2) \mapsto S^1$ by means of

$$\tilde{C}_\alpha(g, r) = (C_\alpha)_{SL}(g)e^{i\pi\mathrm{Tr}(k\epsilon\bar{r}\epsilon)}$$

$\tilde{C}_\alpha$ is an extension of C_α to $(G_\alpha)_{SL} \oplus H(2)$ and a homomorphism. Its differential coincides with the restriction of α to the Lie algebra of this group.

Proposition 4.1. The canonical action of G on $G/\mathrm{Ker}C_\alpha$ maps horizontal orbits to horizontal orbits, thus defining a transitive action on the space of horizontal orbits. The isotropy subgroup at the horizontal orbit of $\mathrm{Ker}C_\alpha$ is $\mathrm{Ker}\tilde{C}_\alpha$.

Proof. Let $(B, R) \in G$, $K \in H(2)$, and let us denote by $((B, R)\cdot)$ and $((I, K)*)$ the diffeomorphisms associated to them by the canonical action and the horizontal action, respectively. We have

$$((B, R)\cdot) \circ ((I, K)*) = ((I, BKB^*)*) \circ ((B, R)\cdot) \tag{6}$$

In fact, for all $(A, H) \in G$ we have

$$
\begin{aligned}
&((B, R)\cdot) \circ ((I, K)*)((A, H)\ \mathrm{Ker}C_\alpha) \\
&\quad = ((B, R)(I, K)(A, H)\ \mathrm{Ker}C_\alpha) \\
&\qquad * e^{-i\pi \mathrm{Tr}(AkA^*\epsilon \bar{K}\epsilon)} \\
&\quad = ((I, BKB^*)(B, R)(A, H)\ \mathrm{Ker}C_\alpha) * e^{i\pi \mathrm{Tr}(AkA^*\epsilon \bar{K}\epsilon)}
\end{aligned}
$$

but, as a consequence of equation (1), we have $\epsilon = B^*\epsilon\bar{B} = \overline{B^*}\epsilon B$, so that

$$
\mathrm{Tr}(AkA^*\epsilon\bar{K}\epsilon) = \mathrm{Tr}(BAk(BA)^*\epsilon\overline{BKB^*}\epsilon)
$$

and equation (6) follows.

That the canonical action of G on $G/\mathrm{Ker}C_\alpha$ maps horizontal orbits to horizontal orbits is an obvious consequence of (6).

An element (A, H) of G is in the isotropy subgroup of the horizontal orbit of $\mathrm{Ker}C_\alpha$ if and only if there exists K in $H(2)$ and (B, R) in G_α such that $(A, H) = (I, K)(B, R)$ and $C_\alpha((B, R)) = \exp[-i\pi\mathrm{Tr}(k\epsilon\bar{K}\epsilon)]$. But this is equivalent to saying that (A, H) is in $(G_\alpha)_{SL} \oplus H(2)$ and $\tilde{C}_\alpha(A, H) = 1$. ■

As a consequence of Proposition 4.1, we identify the space of horizontal orbits with $G/\mathrm{Ker}\tilde{C}_\alpha$.

The canonical maps gives us the following homomorphism of principal S^1-bundles:

$$
\begin{array}{ccc}
G/KerC_\alpha & \longrightarrow & G/Ker\tilde{C}_\alpha \\
\downarrow & & \downarrow \\
G/G_\alpha & \longrightarrow & G/((G_\alpha)_{SL} \oplus H(2)) \approx SL/(G_\alpha)_{SL}
\end{array}
$$

Since quantum states correspond to pseudotensorial functions left invariant by the horizontal action, they will be identified with unrestricted pseudotensorial functions on $G/\mathrm{Ker}\tilde{C}_\alpha$.

5. WAVE FUNCTIONS

In this section we give a way to pass from the quantum states described in the preceding section to a more standard way of looking at quantum processes: wave functions.

A quantum state can be interpreted as giving an "amplitude of probability" to each movement. This interpretation is useful for improving intuition, although the values at different movements are located at different fibers.

As mentioned in the introduction, the idea for obtaining wave functions is to "add up" the "amplitudes of probability" for all movements whose representation in space-time contains each given event. Thus, our first task is to determine this set of movements.

Let the orbit of $\alpha = \{a, k\} \in \underline{G}^*$ be the *movement space* of a given elementary particle and let us choose the *classical state space* to be the orbit of $(0, \alpha) \in H(2) \times \underline{G}^*$. In particular, we assume the existence of a homomorphism C_α as in the preceding section and we use the notation therein.

Our definition of quantum states has been motivated by the requirement that nothing essential in quantum mechanics depend on the fact that the classical states space be the orbit of $(0, \alpha)$ or the orbit of (K, α), with $K \neq 0$. In Remark 5.2 at the end of the present section we explain the consequences of this choice. Since the isotropy subgroup at α is G_α and the isotropy subgroup at $(0, \alpha)$ is $(SL_1 \cap SL_2) \oplus \{0\}$, where $SL_1 = \{A \in SL(2, C): AaA^{-1} = a\}$, $SL_2 = \{A \in SL(2, C): AkA^* = k\}$, the movement space will be identified in the canonical way with G/G_α and state space with $H(2) \oplus (SL/(SL_1 \cap SL_2))$, where $SL(2, \mathbf{C})$ has been denoted simply by SL. The natural map from the classical state space to the movement space thus becomes

$$(H, A\, SL_1 \cap SL_2) \in H(2) \times \frac{SL}{SL_1 \cap SL_2} \mapsto (A, H)G_\alpha \in \frac{SL \oplus H(2)}{G_\alpha}$$

The set consisting of the movements containing the event H is the image under the preceding map of $\{H\} \times SL/(SL_1 \cap SL_2)$. The restriction of that map to that set is injective. We thus see that for each event the corresponding set of movements can be "parametrized" by the same homogeneous space: $SL/(SL_1 \cap SL_2)$.

This is a particular case of the following construction.

Let $\mathcal{L}$ be a closed subgroup of G, and $\mathcal{S}$ the subgroup of SL defined by $\mathcal{S} \oplus \{0\} = \mathcal{L} \cap (SL \oplus \{0\})$. Since $G/(\mathcal{S} \oplus \{0\})$ is canonically diffeomorphic to $H(2) \times (SL/\mathcal{S})$, we can identify these manifolds. When this identification is done, the canonical map from $G/(\mathcal{S} \oplus \{0\})$ onto $G/\mathcal{L}$ becomes

$$(K, A\mathcal{S}) \in H(2) \times \frac{SL}{\mathcal{S}} \mapsto (A, K)\mathcal{L} \in \frac{G}{\mathcal{L}}$$

Notice that the restriction to the subset $\{H\} \times (SL/\mathcal{S})$ is injective. If $\mathcal{H}$ is an invariant closed subgroup of $\mathcal{L}$, $\mathcal{H} \cap (SL \oplus \{0\})$ is an invariant closed subgroup of $\mathcal{L} \cap (SL \oplus \{0\})$.

This geometrical construction, when applied to each of the homogeneous spaces which appear in the commutative diagram of the preceding section, gives us the diagram of Fig. 1.

The vertical arrows correspond to principal fiber bundles whose structural groups are identified by means of $(C_\alpha)_{SL}$, $\tilde{C}_\alpha$, or C_α to subgroups of S^1. The bundle action on $G/\mathrm{Ker}C_\alpha$ is defined by the homomorphism C_α in the

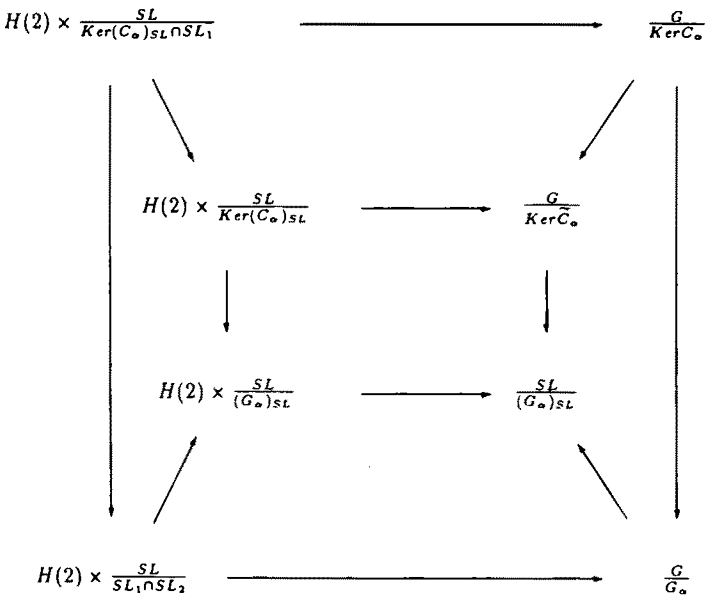


Fig. 1. Fiber bundles for quantum states.

way explained in the preceding section. The other bundle actions are defined by the corresponding homomorphisms in a similar way. The horizontal and oblique arrows define homomorphisms of principal fiber bundles.

The second horizontal arrow will be denoted by ι.

Notice that the energy-momentum, which is defined on movement space (i.e., on G/G_α), is projectable to a function defined on $SL/(G_\alpha)_{SL}$. Its projection will be also denoted by P and is given by $P(A(G_\alpha)_{SL}) = -AkA^*$.

Proposition 5.1. The pullback by ι maps in a one-to-one way the set of quantum states (considered as pseudotensorial functions on $G/\mathrm{Ker}\tilde{C}_\alpha$) onto the set made up of the pseudotensorial functions on $H(2) \times SL/\mathrm{Ker}(C_\alpha)_{SL}$ of the form

$$\begin{aligned}&\phi(H, A\ \mathrm{Ker}(C_\alpha)_{SL})\\ &\quad = f(A\ \mathrm{Ker}(C_\alpha)_{SL}) \exp[i\pi\ \mathrm{Tr}(P(A(G_\alpha)_{SL})\epsilon \bar{H} \epsilon)] \qquad (7)\end{aligned}$$

where f is a pseudotensorial function on the principal fiber bundle $SL/\mathrm{Ker}(C_\alpha)_{SL} \mapsto SL/(G_\alpha)_{SL}$.

Proof. If $B \in (G_\alpha)_{SL}$, there exists $K \in H(2)$ such that $(B, K) \in G_\alpha$. Thus, equation (2) implies $BkB^* = k$. As a consequence, the function defined on $H(2) \times SL/\mathrm{Ker}(C_\alpha)_{SL}$ by

$$\chi(H, A\ \mathrm{Ker}(C_\alpha)_{SL}) = e^{i\pi \mathrm{Tr}(AkA^*\epsilon\overline{H}\epsilon)}$$

is well defined.

Let ϕ' be a pseudotensorial function on $G/\mathrm{Ker}\tilde{C}_\alpha$. We have

$$\begin{aligned}
&((\phi' \circ \iota)\chi)(H, A\ \mathrm{Ker}(C_\alpha)_{SL}) \\
&\quad = \phi'((A, H)\ \mathrm{Ker}\tilde{C}_\alpha * \exp[-i\pi\ \mathrm{Tr}(AkA^*\epsilon\overline{H}\epsilon])) \\
&\quad = \phi'((A, H)\ \mathrm{Ker}\tilde{C}_\alpha * \exp[-i\pi\ \mathrm{Tr}(k\epsilon\overline{A^{-1}HA^{*-1}}\epsilon)]) \\
&\quad = \phi'((A, H)\ \mathrm{Ker}\tilde{C}_\alpha * \tilde{C}_\alpha(I, -A^{-1}HA^{*-1})) \\
&\quad = \phi'((A, H)(I, -A^{-1}HA^{*-1})\ \mathrm{Ker}\tilde{C}_\alpha) \\
&\quad = \phi'((A, 0)\ \mathrm{Ker}\tilde{C}_\alpha)
\end{aligned}$$

We thus see that $((\phi' \circ \iota)\chi)(H, A\ \mathrm{Ker}(C_\alpha)_{SL})$ does not depend on H. Since it is pseudotensorial, it follows that $\phi \equiv \phi' \circ \iota$ has the form (7).

Conversely, let ϕ be a function of the form (7). Notice that the manifold that appears in the preceding diagram as $H(2) \times SL/\mathrm{Ker}(C_\alpha)_{SL}$ was originally identified with $G/(\mathrm{Ker}(C_\alpha)_{SL} \oplus \{0\})$. We consider it under this form, up to the end of the present proof. For all $A \in SL(2, \mathbf{C})$, $a \in (G_\alpha)_{SL}$, H, $h \in H(2)$ we have

$$\begin{aligned}
&\phi((A, H)(a, h)(\mathrm{Ker}(C_\alpha)_{SL} \oplus \{0\})) \\
&\quad = \phi((Aa, AhA^* + H)(\mathrm{Ker}(C_\alpha)_{SL} \oplus \{0\})) \\
&\quad = f(Aa\ \mathrm{Ker}(C_\alpha)_{SL}) \\
&\qquad \times \exp[-i\pi\ \mathrm{Tr}(Aaka^*A^*\epsilon\overline{(AhA^* + H)}\epsilon)] \\
&\quad = f(A\ \mathrm{Ker}(C_\alpha)_{SL} * (C_\alpha)_{SL}(a)) \\
&\qquad \times \exp[-i\pi\ \mathrm{Tr}(AkA^*\epsilon\overline{AhA^*}\epsilon)]\ \exp[-i\pi\ \mathrm{Tr}(AkA^*\epsilon\overline{H}\epsilon)] \\
&\quad = f(A\ \mathrm{Ker}(C_\alpha)_{SL})((C_\alpha)_{SL}(a))^{-1} \\
&\qquad \times \exp[-i\pi\ \mathrm{Tr}(k\epsilon\overline{h}\epsilon)]\ \exp[-i\pi\ \mathrm{Tr}(AkA^*\epsilon\overline{H}\epsilon)] \\
&\quad = \phi((A, H)(\mathrm{Ker}(C_\alpha)_{SL} \oplus \{0\})(\tilde{C}_\alpha((a, h)))^{-1})
\end{aligned}$$

This proves that ϕ maps to a function on $G/\mathrm{Ker}\tilde{C}_\alpha$ and that the projected function is pseudotensorial. ■

Thus, the quantum states are in a one-to-one correspondence with the pseudotensorial functions of the form (7), and these with the pseudotensorial functions on $G/\mathrm{Ker}(C_\alpha)_{SL}$.

Let us denote $(C_\alpha)_{SL}((G_\alpha)_{SL})$ by S, $(H(2) \times (SL/\mathrm{Ker}(C_\alpha)_{SL}) \times_S \mathbf{C}$ by W, and the canonical map from W onto $H(2) \times (SL/(G_\alpha)_{SL})$ by η. Here η is the Hermitian line bundle associated to the principal fiber bundle $(H(2) \times (G/\mathrm{Ker}(C_\alpha)_{SL}))(H(2) \times (G/(G_\alpha)_{SL}), S)$ and the canonical action of S on $\mathbf{C}$.

Proposition 5.1 enables us to interpret quantum states as sections of η. The interesting fact here is that these sections depend on two separate variables, one of them describing an event and the other the set of orbits by $\{\mathbf{I}\} \times H(2)$ of movements containing the event.

To complete our connection to wave functions, we need to "represent" quantum states as functions with values in a fixed complex vector space. If $(C_\alpha)_{SL}$ is trivial, this is done directly since the corresponding fiber bundle is trivial, so that quantum states are identified with complex-valued functions on the base space. In the case where $(C_\alpha)_{SL}$ is not trivial, this will be done by imbedding the Hermitian fiber bundle in a trivial one. We do this in a direct way, but a more geometrical view of the method is given in Remark 5.1.

By a *trivialization of* C_α we mean a triple (ρ, L, z_0), where L is a finite-dimensional complex vector space, $z_0 \in L$, and ρ is a representation of $SL(2, \mathbf{C})$ in L such that:

(1) $\rho(A)(z_0) = (C_\alpha)_{SL}(A)z_0$, $\forall A \in (G_\alpha)_{SL}$.

(2) The isotropy subgroup at z_0 is $\mathrm{Ker}(C_\alpha)_{SL}$.

In what follows, we assume that a trivialization of C_α is given.

The orbit of z_0, $\mathcal{B}$, will be identified with $SL/\mathrm{Ker}(C_\alpha)_{SL}$ and the canonical map onto $SL/(G_\alpha)_{SL}$ will be denoted by $\mathbf{r}$. The pseudotensorial functions on $SL/\mathrm{Ker}(C_\alpha)_{SL}$ thus become functions on $\mathcal{B}$, in fact, they correspond to the functions which are homogeneous of degree -1 under multiplication by elements of $(C_\alpha)_{SL}((G_\alpha)_{SL}) \subset S^1$. These functions will be called α-homogeneous of degree -1. The α-homogeneous functions of degree $-T$ are defined in a similar way.

With this identification, one sees that each α-homogeneous function of degree -1 gives rise to one of the sections of η under consideration.

We define a map δ from W onto $H(2) \times (SL/(G_\alpha)_{SL}) \times L$ by sending the equivalence class of

$$((H, A\ \mathrm{Ker}(C_\alpha)_{SL}), c) \in (H(2) \times (SL/\mathrm{Ker}(C_\alpha)_{SL})) \times \mathbf{C}$$

to $(H, A(G_\alpha)_{SL}, c\rho(A)(z_0))$.

This map is injective and its image will be denoted in what follows by $\mathcal{V}$.

Let us denote by π_1 the restriction to $\mathcal{V}$ of the canonical map from $H(2) \times (SL/(G_\alpha)_{SL}) \times L$ onto $H(2) \times (SL/(G_\alpha)_{SL})$. The image by δ of the fiber over $(H, A(G_\alpha)_{SL})$ consists of the $(H, A(G_\alpha)_{SL}, c\rho(A)(z_0))$ with $c \in \mathbf{C}$. This set also coincides with $(\pi_1)^{-1}(H, A(G_\alpha)_{SL})$. When one considers on $\mathcal{V}$ the topology and differentiable structure which makes δ a diffeomorphism, $\mathcal{V}$ is the total space of a Hermitian line bundle whose bundle projection is π_1 and δ becomes an isomorphism of Hermitian line bundles over the identity of the base manifold. The fibers of π_1 are one-dimensional subspaces of L. The sections of η are thus in a one-to-one correspondence with the sections of π_1, so that each α-homogeneous function of degree -1 defines a section of π_1.

If f is an α-homogeneous function of degree -1, the corresponding section of π_1 sends (H, m) to

$$\delta([(H, z), f(z)e^{i\pi \mathrm{Tr}(P(m)\epsilon\bar{H}\epsilon)}]) = (H, m, f(z)e^{i\pi \mathrm{Tr}(P(m)\epsilon\bar{H}\epsilon)}z)$$

where z is an arbitrary element of $\mathbf{r}^{-1}(m) \subset \mathcal{B}$.

These sections are obviously in a one-to-one correspondence with the functions on $H(2) \times (SL(G_\alpha)_{SL})$ with values in L given by its third components. The quantum state corresponding to the preceding f can thus be identified with the function ψ_F given by

$$\psi_f(H, m) = f(z)e^{i\pi \mathrm{Tr}(P(m)\epsilon\bar{H}\epsilon)}z$$

where $z \in \mathbf{r}^{-1}(m)$. We shall call these functions *prewave functions*.

The same name will be used, in the case in which $(C_\alpha)_{SL}$ is trivial, for the complex-valued functions of the form

$$\psi_f(H, m) = f(m)e^{i\pi \mathrm{Tr}(P(m)\epsilon\bar{H}\epsilon)}$$

where now f is a function on $SL/(G_\alpha)_{SL}$.

The precise meaning of the prescription given at the beginning of this section to obtain wave functions is the following: to any given prewave function ψ_f one can associate a *wave function* $\tilde{\psi}_f$ as follows:

$$\tilde{\psi}_f(x) = \int_{SL/(G_\alpha)SL} \psi_f(h(x), \cdot)\omega$$

where ω is an invariant volume element on $SL(G_\alpha)_{SL}$.

This definition of wave functions forces us to make a restriction on the class of the functions to be considered: it is necessary that the integral exist.

In the following we shall consider only quantum states corresponding to the case in which f is continuous with compact support. The corresponding wave functions are analytic.

Now we shall define a Hermitian product in the vector space consisting of these quantum states. Let ϕ and ϕ' be the quantum states corresponding to the prewave functions ψ_f and $\psi_{f'}$, respectively. Let σ and σ' be the corresponding sections of η. Since

$$\sigma(H, m) = [(H, z), f(z)e^{i\pi \mathrm{Tr}(P(m)\epsilon\bar{H}\epsilon)}]$$

$$\sigma'(H, m) = [(H, z), f'(z)e^{i\pi \mathrm{Tr}(P(m)\epsilon\bar{H}\epsilon)}]$$

where $z \in \mathbf{r}^{-1}(m)$, the Hermitian product of $\sigma(H, m)$ by $\sigma'(H, m)$ is $\overline{f(z)}f'(z)$. In particular, it does not depend on H. This enables us to define the *Hermitian product* of the given quantum states (or the corresponding prewave functions) as being

$$\langle \psi_f, \psi_{f'} \rangle = \int_{SL/(G_\alpha)_{SL}} \bar{f} f' \omega$$

This Hermitian product can be given in terms of the prewave functions themselves as follows. Let Φ be a sesquilinear form on L which does not vanish on $\mathcal{B}$. We define

$$\psi_f \Phi \psi_{f'}: \quad m \in SL/(G_\alpha)_{SL} \mapsto \frac{\Phi(\psi_f(H, m), \psi_{f'}(H, m))}{\Phi(z, z)}$$

where z is arbitrary in $\mathbf{r}^{-1}(m)$ and H is arbitrary in $H(2)$. Thus

$$\langle \psi_f, \psi_{f'} \rangle = \int_{SL/(G_\alpha)_{SL}} \psi_f \Phi \psi_{f'} \omega$$

The well-known fact that the vector space of wave functions is a representation space of the group $SL(2, \mathbf{C}) \oplus H(2)$ can be justified from our present point of view as follows. Since the group $SL(2, \mathbf{C}) \oplus H(2)$ acts on $H(2)$ and on L, there is a canonical representation of the group on the vector space composed of the maps from $H(2)$ into L. The set of wave functions under consideration is an invariant subspace, so that we obtain a representation of the group $SL(2, \mathbf{C}) \oplus H(2)$ in the space of wave functions. Let us denote the infinitesimal generator of this representation associated with X in the Lie algebra of $SL(2, C) \oplus H(2)$ by $\tilde{X}$, and $\tilde{X}/2\pi i$ by $\hat{X}$. Here X is a dynamical variable (see Section 3) and $\hat{X}$ is the corresponding operator in ordinary quantum mechanics.

A direct computation leads to the following expressions for the operators corresponding to the canonical dynamical variables:

$$\hat{p}^k \cdot \tilde{\psi}_f = \frac{1}{2\pi i} \frac{\partial}{\partial x^k} \tilde{\psi}_f$$

$$\hat{p}^4 \cdot \tilde{\psi}_f = \frac{i}{2\pi} \frac{\partial}{\partial x^4} \tilde{\psi}_f$$

$$\hat{l}^k \cdot \tilde{\psi}_f = \frac{1}{2\pi i} \left(\frac{i\sigma_k}{2} + \sum_{j,r=1}^{3} \epsilon_{kjr} x^j \frac{\partial}{\partial x^r} \right) \tilde{\psi}_f$$

$$\hat{g}^k \cdot \tilde{\psi}_f = \frac{1}{2\pi i} \left[\frac{\sigma_k}{2} - \left(x^4 \frac{\partial}{\partial x^k} + x^k \frac{\partial}{\partial x^4} \right) \right] \tilde{\psi}_f$$

where ϵ_{ijk} are the components of an antisymmetric tensor such that $\epsilon_{123} = 1$, and, if $a \in sl(2, C)$, $\tilde{a}$ means the infinitesimal generator of the action on L associated with a, considered as an endomorphism of L.

As we have seen, quantum states can be considered as pseudotensorial functions of different principal fiber bundles, or sections of different Hermitian line bundles, or prewave functions, or wave functions. Under each of these forms, the representation of the group $SL(2, \mathbf{C}) \oplus H(2)$ on quantum states has a natural description.

The following sections are devoted to carrying out explicitly the constructions considered in this section in a number of concrete cases. We take representatives of the coadjoint orbits that leads to what are usually accepted as physically meaningful particles and then we prove that the wave functions we obtain are solutions of the usual wave equations. The representatives we take of the coadjoint orbits are the canonical ones obtained in Díaz Miranda (n.d.).

Remark 5.1. Let L be a finite-dimensional complex vector space and ρ a representation of a Lie group G in L. We denote by $\mathbf{P(L)}$ the projective space of L, i.e., the differentiable manifold of the one-dimensional complex subspaces of L. The representation of G on L induces canonically an action on $\mathbf{P(L)}$ by means of $g * [z] = [\rho(g)\cdot z]$ for all $g \in G$, $z \in L$, where $[z]$ represents the subspace of L generated by z.

If $L^* = L - \{0\}$ and $\mathbf{C}^* = \mathbf{C} - \{0\}$, then $L^*(\mathbf{P(L)}, \mathbf{C}^*)$, is a principal fiber bundle whose bundle projection is the canonical map defined by sending each nonzero element of L to the subspace of L it generates.

Let η: $L^* \times_{\mathbf{C}^*} C \mapsto \mathbf{P(L)}$ be the line bundle associated with this principal fiber bundle and the canonical action of $\mathbf{C}^*$ on $\mathbf{C}$. The total space of this vector bundle can be immersed in $\mathbf{P(L)} \times L$ by means of the map i defined by $i([l, c]) = ([l], cl)$. Of course, the total space can also be considered as L^* "completed with the zero section" in the sense that the map from $L^* \cup \mathbf{P(L)}$ onto $L^* \times_{\mathbf{C}^*} C$ defined by sending $l \in L^*$ to $[l, 1]^*_{\mathbf{C}}$ and $[l] \in \mathbf{P(L)}$ to $[l, 0]^*_{\mathbf{C}}$ is bijective.

Now let $z_0 \in L^*$, G_{z_0} the isotropy subgroup at z_0, and $G_{[z_0]}$ the isotropy subgroup at $[z_0]$. Thus $G_{[z_0]}$ consists of the $g \in G$ such that $\rho(g)\cdot z_0 = \lambda z_0$ for some $\lambda \in \mathbf{C}^*$. If such λ is denoted by $K(g)$, we obtain a homomorphism K from $G_{[z_0]}$ into $\mathbf{C}^*$. The isotropy subgroup at z_0 is the kernel of K.

Let H be a closed subgroup of $G_{[z_0]}$ that contains G_{z_0}. We identify H/G_{z_0} with $K(H)$ as groups in the canonical way, but we maintain the quotient topology and differentiable structure of H/G_{z_0}. Now, $G/G_{z_0}(G/H, K(H))$ is a principal fiber bundle and the maps h_0: $gG_{z_0} \in G/G_{z_0} \mapsto \rho(g)\cdot z_0 \in L^*$ and h: $gH \in G/H \mapsto [\rho(g)\cdot z_0] \in \mathbf{P(L)}$, whose images are the orbits of z_0 and $[z_0]$, respectively, define a homomorphism of principal fiber bundles into

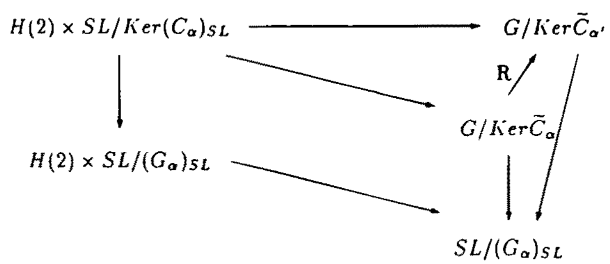
$L^*(\mathbf{P}(\mathbf{L}), \mathbf{C}^*)$. One can immerse $(G/G_{z_0}) \times_{K(H)} \mathbf{C}$ in $G/H \times L$ in the same way that $L^* \times_{\mathbf{C}^*}$ has been immersed in $\mathbf{P}(\mathbf{L}) \times L$: an immersion τ is defined by sending each $[gG_{z_0}, c]$ to $(gH, c\rho(g)\cdot z_0)$.

On the other hand, there is a canonical action of G on $(G/G_{z_0}) \times_{K(H)} \mathbf{C}$ given by $g * [fG_{z_0}, c]_{K(H)} = [gfG_{z_0}, c]_{K(H)}$. This action and the canonical one on G/H provide us with a representation of G on the vector space of sections of the line bundle ν: $(G/G_{z_0}) \times_{K(H)} \mathbf{C} \mapsto G/H$. But τ gives an injective map from sections of ν into functions on G/H with values in L: if σ is a section of ν, the composite map $\tau \circ \sigma$ is a map of the form $u \mapsto (u, y(u))$, so that the map defined by sending σ to y is injective. By means of this injective map, we transform our representation in a space of sections of an in general nontrivial bundle to a representation in a space of functions with values in a fixed vector space.

Our definition of trivialization is such that $G_{z_0} = \mathrm{Ker}(C_\alpha)_{SL}$, $G_{z_0} \subset (G_\alpha)_{SL} \subset G_{[z_0]}$ and the restriction of K to $(G_\alpha)_{SL}$ coincides with $(C_\alpha)_{SL}$. The geometrical construction we have done in this remark thus gives rise to the results stated in the main body of this section.

Remark 5.2. If we choose the classical state space to be the orbit of (K, α) with $K \neq 0$ instead of the orbit corresponding to $K = 0$, the wave functions we obtain are the same. In fact, the orbit of (K, α) is the orbit of $(0, \mathrm{Ad}^*_{(I,-K)}\alpha)$. Let us denote $\mathrm{Ad}^*_{(I,-K)}\alpha$ by α'. We have $G_{\alpha'} = a_{(I,-K)}(G_\alpha)$, where $a_{(I,-K)}$ is the internal automorphism corresponding to $(I, -K)$, i.e., $a_{(I,-K)}((A, H)) = (I, -K)(A, H)(I, K)$. The form α' is also quantizable with $C_{\alpha'} = C_\alpha \circ a_{(I,K)}$, and we obtain by straightforward computations $\mathrm{Ker}C_{\alpha'} = a_{(I,-K)}(\mathrm{Ker}C_\alpha)$, $(G_{\alpha'})_{SL} = (G_\alpha)_{SL}$, $(C_{\alpha'})_{SL} = (C_\alpha)_{SL}$ [use equation (1) to prove that $\mathrm{Tr}(k\epsilon\overline{(AKA^* - K)}\epsilon) = 0$ if $A \in (G_\alpha)_{SL}$], $\tilde{C}_{\alpha'} = \tilde{C}_\alpha \circ a_{(I,K)}$, and $\mathrm{Ker}\tilde{C}_{\alpha'} = a_{(I,-K)}(\mathrm{Ker}\tilde{C}_\alpha)$.

We thus have the following commutative diagram:



where R is given by $R(V\,\mathrm{Ker}\tilde{C}_\alpha) = V(I, K)\,\mathrm{Ker}\tilde{C}_{\alpha'}$ and defines an isomorphism of principal fiber bundles. Thus R establishes a one-to-one map from pseudotensorial functions to pseudotensorial functions. Two corresponding pseudotensorial functions gives rise to the same wave function.

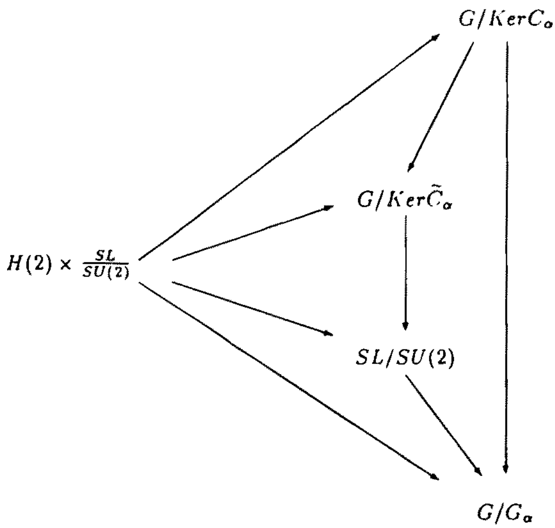
6. MASSIVE PARTICLES

Let us consider a particle whose movement space is the coadjoint orbit of

$$\alpha = \{0, \eta mI\}, \qquad m \in \mathbf{R}^+, \quad \eta = \pm 1$$

This orbit is an **R**-quantizable orbit of the type 5, in the notation of Díaz Miranda (n.d.). Here $(-\eta)$ is the sign of energy, i.e., the sign of the value of the dynamical variable P^4 (see Section 3) at any point of the orbit.

By direct computation, one sees that $G_\alpha = \{(A, hI): A \in SU(2), h \in \mathbf{R}\}$. The unique homomorphism onto **R** whose differential is α is given by $C'_\alpha(A, hI) = -\eta mh$. The unique homomorphism onto $\mathbf{S}^1$ whose differential is α is given by $C_\alpha(A, hI) = e^{-2\pi i\eta mh}$. Then we have $\tilde{C}_\alpha(A, H) = e^{-\pi i\eta m \operatorname{Tr} H}$, $(G_\alpha)_{SL} = SU(2)$, $(C_\alpha)_{SL} = \mathbf{1}$, $SL_1 = SL_2 = \operatorname{Ker}(C_\alpha)_{SL} = (G_\alpha)_{SL}$, so that in the commutative diagram of Fig. 1 the four spaces on the left are the same. Thus the diagram becomes



Let $\mathcal{H}^m$ be the mass hyperboloid

$$\mathcal{H}^m = \{H \in H(2): \det H = m^2, \operatorname{Tr} H > 0\}$$

In $\mathcal{H}^m$ we consider the action of $SL(2, C)$ given by $A * H = AHA^*$. This action is transitive. The isotropy subgroup at mI is $SU(2)$, so that $SL/(G_\alpha)_{SL}$ can be identified with $\mathcal{H}^m$. The function P thus becomes $P(K) = -\eta K$ for all $K \in \mathcal{H}^m$.

An invariant volume element on $\mathcal{H}^m$ is given by

$$\nu = \frac{dp^1 \wedge dp^2 \wedge dp^3}{[m^2 + \sum_{i=1}^3 (p^i(K))^2]^{1/2}}$$

where the p^i are the coordinates corresponding to the following parametrization of $\mathcal{H}^m$:

$$(p^1, p^2, p^3) \in \mathbf{R}^3 \mapsto h(p^1, p^2, p^3, (m^2 + \textstyle\sum_{i=1}^3 (p^i)^2)^{1/2}) \in \mathcal{H}^m$$

Since $(C_\alpha)_{SL}$ is trivial, we need no trivialization in this case. The prewave functions have the form

$$\psi_f\colon \quad (H, K) \in H(2) \times \mathcal{H}^m \mapsto f(K)e^{-i\pi\eta \mathrm{Tr}(K\epsilon \bar{H}\epsilon)}$$

where f is a continuous function on $\mathcal{H}^m$ with compact support.

The corresponding wave function is

$$\tilde{\psi}_f(X) = \int_{\mathcal{H}^m} \psi_f(h(X), \cdot)\nu$$

By direct computation one sees that these wave functions satisfy the *Klein–Gordon* equation.

If f' is another function continuous with compact support on $\mathcal{H}^m$, the Hermitian product of the quantum states corresponding to ψ_f and $\psi_{f'}$ can be written

$$\langle \psi_f, \psi_{f'} \rangle = \int_{\mathcal{H}^m} \psi_f^* \psi_{f'} \nu$$

Now let us consider a particle whose movement space is the coadjoint orbit of

$$\alpha = \left\{ \frac{iT}{8\pi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \eta m I \right\} \tag{8}$$

where $T \in \mathbf{Z}^+$, $m \in \mathbf{R}^+$, $\eta = \pm 1$. This orbit is a quantizable, not $\mathbf{R}$-quantizable orbit, of type 5 in the notation of Díaz Miranda (n.d.).

In this case we have

$$G_\alpha = \left\{ \left(\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}, hI \right) : z \in S^1, h \in \mathbf{R} \right\}$$

The unique homomorphism from G_α onto $\mathbf{S}^1$ whose differential is α is given by

$$C_\alpha\left(\begin{pmatrix} e^{2\pi i\phi} & 0 \\ 0 & e^{-2\pi i\phi} \end{pmatrix}, hI\right) = e^{2\pi i(\phi T - \eta mh)}$$

Then

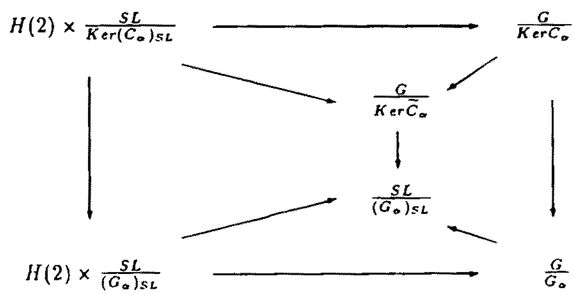
$$(G_\alpha)_{SL} = \left\{\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} : z \in S^1\right\}$$

$$(C_\alpha)_{SL}\left(\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}\right) = z^T$$

$$SL_1 \cap SL_2 = (G_\alpha)_{SL}$$

$$SL_1 \cap \operatorname{Ker}(C_\alpha)_{SL} = \operatorname{Ker}(C_\alpha)_{SL}$$

The commutative diagram becomes in this case



The homogeneous space $SL/(G_\alpha)_{SL}$ can be characterized as follows.

Let $\mathbf{P}_1(\mathbf{C})$ be the complex projective space corresponding to $\mathbf{C}^2$ [i.e., $\mathbf{P}_1(\mathbf{C})$ consists of the one-dimensional complex subspaces of $\mathbf{C}^2$].

In $\mathcal{H}^m \times \mathbf{P}_1(\mathbf{C})$ one can consider the action of $SL(2, C)$ given by

$$A * (H, [z]) = (AHA^*, [Az])$$

where $[z] \in \mathbf{P}_1(\mathbf{C})$ is the equivalence class of $x \in \mathbf{C}^2$.

The isotropy subgroup at $(mI, [(\begin{smallmatrix}1\\0\end{smallmatrix})])$ is $(G_\alpha)_{SL}$, so that, since the action is transitive, one can identity $SL/(G_\alpha)_{SL}$ with $\mathcal{H}^m \times \mathbf{P}_1(\mathbf{C})$.

Let us consider the 5-form in $\mathcal{H}^m \times \mathbf{C}^2$ given by

$$(\mu_0)_{(K,z)} = \nu \wedge \frac{(z^1dz^2 - z^2dz^1) \wedge \overline{(z^1dz^2 - z^2dz^1)}}{(z^*\epsilon\overline{K}\epsilon z)^2}$$

where the z^k are the two canonical projections of $\mathbf{C}^2$ onto C.

This differential form projects to an invariant volume element μ in $\mathcal{H}^m \times \mathbf{P}_1(\mathbf{C})$.

In order to describe prewave functions in the case $T = 1$, one can consider the trivialization $(\rho, \mathbf{C}^4, z_0)$, where z_0 is the transpose of (1, 0, 1, 0) and ρ is given by

$$\rho(A) = \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$$

The orbit $\mathcal{B}$ of z_0 is

$$\mathcal{B} = \left\{ \begin{pmatrix} w \\ z \end{pmatrix} : w, z \in \mathbf{C}^2, z^*w = 1 \right\} \tag{9}$$

When one identifies $SL/\text{Ker}(C_\alpha)$ with $\mathcal{B}$ and $SL/(G_\alpha)_{SL}$ with $\mathcal{H}^m \times \mathbf{P}_1(\mathbf{C})$ by means of the preceding actions the canonical map between these homogeneous spaces becomes a map $\mathbf{r}$ from $\mathcal{B}$ onto $\mathcal{H}^m \times \mathbf{P}_1(\mathbf{C})$. By direct computation one sees that this map is given by

$$\mathbf{r}\begin{pmatrix} w \\ z \end{pmatrix} = (m(ww^* - \epsilon\overline{zz^*}\epsilon), [w])$$

Also we have

$$\mathbf{r}^{-1}(K, [a]) = \left\{ s\begin{pmatrix} I \\ mK^{-1} \end{pmatrix} \frac{a}{(ma^*K^{-1}a)^{1/2}} : s \in \mathbf{S}^1 \right\}$$

$$P(K, [a]) = -\eta K$$

If f is a function on $\mathcal{B}$ which is continuous, has compact support, and is homogeneous of degree -1 under multiplication by complex numbers of modulus one (i.e., α-homogeneous of degree -1), the corresponding prewave function is

$$\psi_f\colon \quad (H, K, [a]) \in H(2) \times \mathcal{H}^m \times \mathbf{P}_1(\mathbf{C}) \mapsto f\begin{pmatrix} w \\ z \end{pmatrix} e^{-i\pi\eta \operatorname{Tr}(K\epsilon\overline{H}\epsilon)} \begin{pmatrix} w \\ z \end{pmatrix}$$

where $\binom{w}{z}$ is arbitrary in $\mathbf{r}^{-1}(K, [a])$.

The corresponding wave function is

$$\tilde{\psi}_f(X) = \int_{\mathcal{H}^m \times P_1(C)} \psi_f(h(X), \cdot, \cdot)\mu$$

When one considers the Dirac matrices in the representation

$$\gamma^4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \qquad k = 1, 2, 3$$

one sees that these wave functions satisfy the Dirac equation

$$(\gamma^\nu \partial_\nu - 2\pi i \eta m)\tilde{\psi}_f = 0$$

Let f' be another function on $\mathcal{B}$ which, like f, is continuous, has compact support, and is homogeneous of degree -1 under multiplication by complex numbers of modulus one. A sesquilinear form on $\mathbf{C}^4$ whose value on $\mathcal{B}$ is 1 is defined by

$$\Phi(Z, Z') = \tfrac{1}{2}Z^*\gamma^4 Z$$

Thus, the Hermitian product of the quantum states corresponding to ψ_f and $\psi_{f'}$ (see Section 6) can be written

$$\langle \psi_f, \psi_{f'} \rangle = \frac{1}{2}\int_{\mathcal{H}^m \times P_1(C)} \psi_f^* \gamma^4 \psi_{f'} \mu$$

In order to give the wave functions in the case where $T > 1$, the following results are useful.

Let α, α' be quantizable elements of $\underline{G}^*$ such that α' is not $\mathbf{R}$-quantizable, $(G_\alpha)_{SL} = (G_{\alpha'})_{SL}$, and $(C_\alpha)_{SL} = ((C_{\alpha'})_{SL})^T$, where $T \in \mathbf{Z}^+$. If (ρ, L, z_0) is a trivialization of $C_{\alpha'}$, we consider the triple $(\rho^{\otimes T}, L^{\otimes T}, z_0^{\otimes T})$, where

$$L^{\otimes T} = L \otimes \overset{(T}{\cdots} \otimes L, \qquad z_0^{\otimes T} = z_0 \otimes \overset{(T}{\cdots} \otimes z_0$$

and $\rho^{\otimes T}$ is the representation such that

$$\rho^{\otimes T}(A)(z_1 \otimes \cdots \otimes z_T) = \rho(A)(z_1) \otimes \cdots \otimes \rho(A)(z_T)$$

Let us assume that $(\rho^{\otimes T}, L^{\otimes T}, z_0^{\otimes T})$ is a trivialization of C_α and let $\mathcal{B}_T$ be the orbit of $z_0^{\otimes T}$. The pullback by $z \in \mathcal{B} \to z^{\otimes T} \in \mathcal{B}_T$ establishes a one-to-one map from the set of the α-homogeneous functions of degree -1 onto the set of the α'-homogeneous functions of degree $-T$. If f is one of these functions, the corresponding prewave function of particles corresponding to α has the form

$$\psi_f(H, m) = f(z)e^{i\pi \operatorname{Tr}(P(m)\epsilon \bar{H} \epsilon)} z^{\otimes T}$$

where $z \in \mathbf{r}^{-1}(m)$.

Lemma 6.1. If $(G_\alpha)_{SL}$ is connected, $(\rho^{\otimes T}, L^{\otimes T}, z_0^{\otimes T})$ is a trivialization of C_α.

Proof. The only nontrivial fact is that the isotropy subgroup at $z_0^{\otimes T}$ is contained in $\operatorname{Ker} C_\alpha$.

We shall first prove that $(C_{\alpha'})_{SL}$ is surjective. In fact, since $\dim \mathbf{S}^1 = 1$, if $(C_{\alpha'})_{SL}$ is not surjective we have $d(C_{\alpha'})_{SL} = 0$. Since $(G_{\alpha'})_{SL}$ is connected, it follows that $(C_{\alpha'})_{SL} = 1$. Thus $C_{\alpha'}(g, r) = \exp[i\pi \operatorname{Tr}(k\epsilon\bar{r}\epsilon)]$ for all $(g, r) \in G_{\alpha'}$, where k is given by $\alpha' = \{b, k\}$. Then the map K from $G_{\alpha'}$ onto $\mathbf{R}$ given by $K(g, r) = (1/2) \operatorname{Tr}(k\epsilon\bar{r}\epsilon)$ is a homomorphism whose differential is the same as the differential of $C_{\alpha'}$, i.e., α'. But this is contradictory with the hypothesis that α' is not **R**-quantizable.

Let A be in the isotropy subgroup at $z_0^{\otimes T}$. Then there exists $d \in \mathbf{C}$ such that $\rho(A)(z_0) = dz_0$ and $d^T = 1$. Since $(C_{\alpha'})_{SL}$ is surjective, there exists $B \in (G_{\alpha'})_{SL}$ such that $(C_{\alpha'})_{SL}(B) = d$. Thus $\rho(A)(z_0) = \rho(B)(z_0)$, so that $B^{-1}A \in \operatorname{Ker}(C_{\alpha'})_{SL}$. Therefore, $A \in (G_{\alpha'})_{SL}$, $(C_{\alpha'})_{SL}(A) = d$, and $(C_\alpha)_{SL}(A) = d^T = 1$. ∎

The same result holds under other hypotheses. In fact, if α' is quantizable and the cohomology class of the restriction of α' to the connected component of the identity of $G_{\alpha'}$ is not zero, α' is not **R**-quantizable (see Díaz Miranda, n.d., Section 8) and a slight modification of the preceding argument proves the same result even though $(G_{\alpha'})_{SL}$ is not connected.

In the case where α is given by (8) and α' is the particular case corresponding to $T = 1$, the preceding lemma applies and one is led to the following prewave functions:

$$\psi_f\colon\quad (H, K, [a]) \in H(2) \times \mathcal{H}^m \times \mathbf{P}_1(\mathbf{C}) \mapsto f\begin{pmatrix} w \\ z \end{pmatrix} e^{-i\pi\eta \operatorname{Tr}(K\epsilon\overline{H}\epsilon)} \begin{pmatrix} w \\ z \end{pmatrix}^{\otimes T}$$

where $\binom{w}{z}$ is arbitrary in $\mathbf{r}^{-1}(K, [a])$ and f is a function on $\mathcal{B}$, continuous, with compact support, and homogeneous of degree $-T$ under multiplication by complex numbers of modulus one.

The wave functions are obtained by integration as usual.

7. MASSLESS PARTICLES

In this section we consider particles whose movement space is the coadjoint orbit of

$$\alpha = \left\{ \frac{i\chi T}{8\pi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \eta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \qquad \chi, \eta \in \{\pm 1\}, \quad T \in \mathbf{Z}^+$$

These are quantizable, not **R**-quantizable orbits of type 4 in the notation of Díaz Miranda (n.d.).

Also in this case, G_α is connected, so that there exists a unique homomorphism from G_α onto $\mathbf{S}^1$ whose differential is α. In fact, we have

$$G_\alpha = \left\{ \left(\begin{pmatrix} z & a \\ 0 & \bar{z} \end{pmatrix}, \begin{pmatrix} b & i\chi\eta T a z/2\pi \\ \overline{i\chi\eta T a z/2\pi} & 0 \end{pmatrix} \right) : z \in \mathbf{S}^1, b \in \mathbf{R}, a \in \mathbf{C} \right\}$$

$$C_\alpha\left(\left(\begin{pmatrix} z & a \\ 0 & \bar{z} \end{pmatrix}, \begin{pmatrix} b & i\chi\eta T a z/2\pi \\ \overline{i\chi\eta T a z/2\pi} & 0 \end{pmatrix} \right) \right) = z^{\chi T}$$

$$(G_\alpha)_{SL} = \left\{\begin{pmatrix} z & a \\ 0 & \bar{z} \end{pmatrix}: z \in \mathbf{S}^1, a \in \mathbf{C}\right\}$$

$$(C_\alpha)_{SL}\left(\begin{pmatrix} z & a \\ 0 & \bar{z} \end{pmatrix}\right) = z^{\chi T}$$

$$SL_1 = \left\{\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}: a \in \mathbf{C}\right\}$$

$$SL_2 = (G_\alpha)_{SL}$$

$$SL_1 \cap SL_2 = \left\{\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}: z \in \mathbf{S}^1\right\}$$

The isotropy subgroup at $(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$ for the usual action of $SL(2, C)$ on $H(2)$ is $(G_\alpha)_{SL}$. Thus $SL/(G_\alpha)_{SL}$ will be identified with the orbit of $(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$, i.e., the future half-cone

$$\mathbf{C}^+ = \{H \in H(2)\colon \text{Det } H = 0, \text{Tr } H > 0\}$$

When this identification is made, the function P on $SL/(G_\alpha)_{SL}$ becomes $P(H) = -\eta H$.

An invariant volume element in $\mathbf{C}^+$ is

$$\omega = \frac{1}{[\Sigma_{i=1}^3 (p^i)^2]^{1/2}}\, dp^1 \wedge dp^2 \wedge dp^3$$

where (p^1, p^2, p^3) is the coordinate system corresponding to the parametrization

$$(p^1, p^2, p^3) \in \mathbf{R}^3 - \{0\} \mapsto h\left(p^1, p^2, p^3, \left[\sum_{i=1}^{3} (p^i)^2\right]^{1/2}\right) \in \mathbf{C}^+$$

Before proceeding to the study of the general use, we consider two particular ones.

First we consider the case in which $T = 1, \chi = 1, \eta = -1$. A trivialization is given by $(\rho, \mathbf{C}^2, (\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}))$, where $\rho(A)$ is multiplication by A.

The orbit of $(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix})$ is $\mathbf{C}^2 - \{0\}$. This space can thus be identified with $SL/\text{Ker}(C_\alpha)_{SL}$, so that one obtains a natural map $r_+\colon \mathbf{C}^2 - \{0\} \mapsto \mathbf{C}^+$. This map is explicitly given by $r_+(z) = zz^* \in \mathbf{C}^+ \subset H(2)$. Since z and $\epsilon\bar{z}$ are eigenvectors of zz^* corresponding to the eigenvalues $\|z\|^2$ and 0, respectively, $r_+^{-1}(H)$ is composed of the eigenvectors of H corresponding to the positive eigenvalue whose norm is that eigenvalue.

The principal $\mathbf{S}^1$-bundle whose projection is r_+ is related to the Hopf fibration as follows. The image of the restriction of r_+ to the sphere $S^3(R)$

$= \{z \in \mathbf{C}^2: \|z\|^2 = R^2\}$ is composed of the elements of $\mathbf{C}^+$ whose trace is R^2. Since the image of this subset by the preceding chart is the sphere of radius $R^2/2$, we obtain maps from spheres S^3 onto spheres S^2. Each one of these mappings is, up to the radius and a reflection, the Hopf fibration.

The prewave functions in this case have the form

$$\psi_f^+: \quad (N, H) \in \mathbf{C}^+ \times H(2) \mapsto f(z) e^{i\pi \operatorname{Tr}(N\epsilon \bar{H}\epsilon)} z \in \mathbf{C}^2$$

where z is an arbitrary element of $r_+^{-1}(N)$.

By direct computation one can see that

$$\left(\sigma_1 \frac{\partial}{\partial x^1} + \sigma_2 \frac{\partial}{\partial x^2} + \sigma_3 \frac{\partial}{\partial x^3} + \sigma_4 \frac{\partial}{\partial x^4}\right)\psi_f^+(N, h(x)) = 0$$

The corresponding wave functions thus satisfy the same equation, which is the *antineutrino* Weyl equation (positive energy).

If f and f' are pseudotensorial functions, we have

$$\psi_f(N, H)^* \psi_{f'}(N, H) = f(z)^* f'(z) \operatorname{Tr} N$$

Thus, the Hermitian product of the corresponding quantum states (see Section 5) can be written as follows:

$$\frac{1}{2} \int_{\mathbf{C}^+} \frac{\psi_f^* \psi_{f'}}{\sum_{i=1}^3 (p^i)^2} \, dp^1 \, dp^2 \, dp^3$$

Now let us consider the case in which $T = 1$, $\chi = -1$, $\eta = -1$. One can use a trivialization similar to the preceding one, $\rho(A)$ being multiplication by $(A^*)^{-1}$, thus obtaining a principal $\mathbf{S}^1$-bundle $\mathbf{C}^2 - \{0\} \mapsto \mathbf{C}^+$, where $r_-(z) = -\epsilon \overline{z z^*} \epsilon \in \mathbf{C}^+ \subset H(2)$. Since z and $\epsilon \bar{z}$ are eigenvectors of $-\epsilon \overline{z z^*} \epsilon$ corresponding to the eigenvalues 0 and $\|z\|^2$, respectively, $r_-^{-1}(H)$ is composed of the elements of the kernel of H whose norm is its positive eigenvalue.

The wave functions one obtains in this case satisfy the Weyl equation that, according to Feynman, corresponds to the *neutrino*.

The map from the real vector space $\mathbf{C}^2$ onto itself defined by sending z to $\epsilon \bar{z}$ is a complex structure and its restriction to $\mathbf{C}^2 - \{0\}$ gives us an isomorphism of the principal circle bundle corresponding to $\mathbf{r}_-$ (resp. $\mathbf{r}_+$) onto the principal circle bundle corresponding to $\mathbf{r}_+$ (resp. $\mathbf{r}_-$). The isomorphism of the structural group is defined by sending each element to its inverse.

The results stated at the end of the preceding section enable us to describe the general case as follows.

The prewave functions are given by functions on $\mathbf{C}^2 - \{0\}$ which are continuous with compact support and homogeneous of degree $-T$ under multiplication by modulus-one complex numbers. Let f_T be one of these functions. If $\chi = 1$, the corresponding prewave function is given by

$$\psi^{+}_{f_T}\colon \quad (N, H) \in \mathbf{C}^{+} \times H(2) \mapsto f_T(z)e^{-i\pi\eta \operatorname{Tr}(N\epsilon\overline{H}\epsilon)} z^{\otimes T} \in (\mathbf{C}^2)^{\otimes T}$$

where z is an arbitrary element of $r_{+}^{-1}(N)$.

In the case $\chi = -1$, the corresponding prewave function is

$$\psi^{-}_{f_T}\colon \quad (N, H) \in \mathbf{C}^{+} \times H(2) \mapsto f_T(z)e^{-i\pi\eta \operatorname{Tr}(N\epsilon\overline{H}\epsilon)} z^{\otimes T} \in (\mathbf{C}^2)^{\otimes T}$$

but now z is an arbitrary element of $r_{-}^{-1}(N)$.

The associated wave functions satisfy Penrose's wave equations, which we describe here for the sake of completeness.

Let us consider in $(\mathbf{C}^2)^{\otimes T}$ the basis

$$\{e_A \otimes e_B \otimes \overset{(T}{\cdots}\colon A, B, \ldots \in \{1, 2\}\}$$

where $\{e_1, e_2\}$ is the canonical basis of $\mathbf{C}^2$.

The prewave functions $\psi^{\pm}_{f_T}$ and the wave functions $\tilde{\psi}^{\pm}_{f_T}$ have components in this basis which will be denoted by $\{\psi^{AB\cdots}_{\pm}\}$ and $\{\tilde{\psi}^{AB\cdots}_{\pm}\}$, respectively.

Let us consider the vector fields in $\mathbf{R}^4$ given by

$$\nabla_{11} = \frac{1}{2}\left(\frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^4}\right)$$

$$\nabla_{12} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - i\,\frac{\partial}{\partial x^2}\right)$$

$$\nabla_{21} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\,\frac{\partial}{\partial x^2}\right)$$

$$\nabla_{22} = \frac{1}{2}\left(\frac{\partial}{\partial x^4} - \frac{\partial}{\partial x^3}\right)$$

and, for all $A, A' \in \{1, 2\}$,

$$\nabla^{AA'} = \epsilon^{AB}\epsilon^{A'B'}\nabla_{BB'}$$

(summation convention), where $\{\epsilon^{AB}\}$ are the elements of $-\epsilon$.

We also define

$$\psi^{\pm}_{AB\ldots} = \epsilon_{AA'}\epsilon_{BB'}\cdots\,\psi^{A'B'\ldots}_{\pm}$$
$$\tilde{\psi}^{\pm}_{AB\ldots} = \epsilon_{AA'}\epsilon_{BB'}\cdots\,\tilde{\psi}^{A'B'\ldots}_{\pm}$$

where $\{\epsilon_{AB}\}$ are the elements of ϵ.

Thus we have for all $h(x) \in C^{+}$

$$\nabla^{AA'}\psi^{+}_{A'BC\ldots}(h(x), \cdot) = 0$$

$$\nabla^{A'A}\psi^{-}_{A'BC\ldots}(h(x), \cdot) = 0$$

so that, by derivation under the integral sign, we see that Penrose wave equations

$$\nabla^{AA'}\tilde{\psi}^{+}_{A'BC\ldots} = 0$$

$$\nabla^{A'A}\tilde{\psi}^{-}_{A'BC\ldots} = 0$$

are satisfied.

ACKNOWLEDGMENT

This work was partially supported by a grant from the DGICYT.

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